MODELLING THE EXPONENTIAL DAMPING OF A PENDULUM

1. INTRODUCTION:

The idea of exponential relationships being so abundant in nature is one I find quite intriguing. The main cause for the commonality of exponential relationships is the ability of systems to be modelled using second order differential equations, as is the case in the damping of a pendulum. In exploring the theory for this investigation, I was fascinated by the involvement of complex numbers in the solution of a system that was real, and the whole aspect of it leading to this relationship. I was also interested in the modelling of oscillatory phenomena which find applications in a wide range of physical systems; pendulums are a model used to understand Josephson Junctions between superconductors, for instance. My interest in both these coincided and led me to decide to investigate damping in pendula. The *damping* of a pendulum can be defined as an effect that results in a reduction of the amplitude of oscillation in a system.

RESEARCH QUESTION: *How does the amplitude of an oscillating pendulum vary with the number of oscillations that have taken place?*

2. BACKGROUND INFORMATION:

a. Exponential Decay in Damping – Justification

In the case of pendula, this is preliminarily a result of viscous drag – a resistant force provided by the medium fluid, as a result of an object moving through it. For spherical objects moving at relatively low speeds, where air flow is laminar, this force is proportional to the velocity of the object, in the opposite direction. This is modelled by Stokes' law[1], which is as follows:

$$F_{drag} = -6\pi\eta R v \qquad (1)$$

Where η is the viscosity of the fluid, R is the radius of the sphere, and v is the velocity of the moving object. The negative sign indicates that the drag is in opposite in direction to the velocity of the object. For the purposes of solving for the damping relationship, the $6\pi\eta R$ term can be said to be a constant *c*. In order to model this oscillating system, a differential equation can be created:

$$ma = kx - cv \qquad (2)$$

Here, *k* refers to the proportionality constant between the restoring force and displacement from equilibrium position (*x*), that is characteristic of simple harmonic motion. The acceleration of the bob is represented by *a.* This equation can be rewritten as follows:

$$m\ddot{x} = kx - c\dot{x} \qquad (3)$$

[1] "Dropping the Ball (Slowly)." *Stokes' Law,* galileo.phys.virginia.edu/classes/152.mf1i.spring02/Stokes_Law.htm

The derivatives here are functions of time *t*. Solving the equation for *x* is an interesting process that involves the use of complex numbers in a trial exponential function. The real part of the final solution of the differential equation is an exponential function of time can be written as follows:

$$x = Ae^{-bt}\cos(\omega t + \alpha) \tag{4}$$

A refers to the initial amplitude of oscillation, *b* is a damping constant, and the cosine function is indicative of the variation of amplitude with time, where ω refers to the angular frequency of oscillation. In reality, ω is a function of the damping coefficient as well, but this investigation will be ignoring this effect since it measures amplitude as a function of the oscillation number (N). A full proof of this solution satisfying the given differential equation is provided in Appendix 1. Thus, the exponential relation can be written as:

$$x = Ae^{-bN} \tag{5}$$

Clearly, *b* is not equal to that in the equation 4 – however this is only a dimensionless constant, and is here reattributed.

b. Calculations Involved

The decay in amplitude be calculated using measurements of the maximum velocity attained by an oscillating bob, which occurs at the equilibrium position of the pendulum. This relation is a consequence of the conservation of energy in a pendulum. The total energy of a pendulum can be expressed as follows:

$$E_{total} = \frac{1}{2}m\omega^2 x_0^2 \tag{6}$$

Here, *m* is the mass of the bob, ω the angular frequency, and x_0 the maximum displacement (amplitude). At the equilibrium position, the bob has no potential energy in context of the pendulum, and all energy exists as kinetic energy – hence velocity is maximum. Thus, the following relation arises:

$$\frac{1}{2}mv_{max}^2 = \frac{1}{2}m\omega^2 x_0^2 \tag{8}$$

Therefore, it can be derived that:

$$x_0 = \frac{v_{max}}{\omega} \tag{9}$$

This investigation measures the time blocked by the bob as it moves through the equilibrium position. If *t* is this time (seconds), and *d* (meters) the diameter of the bob, then based on $v_{max} = \frac{d}{t}$:

$$x_0 = \frac{d}{\omega t} \tag{10}$$

Hence x_0 can be calculated from *d, t,* and ω. The formula for the following is developed below:

As explained in Appendix 1, the decrease in angular frequency is negligible as opposed to the decay in amplitude. Hence, this value is assumed to be constant. Given the small angle used (5°), it is acceptable to use define ω on the basis of the small angle approximation. Although there is a very small discrepancy with the true value of ω, this can be ignored due to this only producing a systematic displacement of values, which does not affect the confirmation of an exponential decay.

$$\omega = \sqrt{\frac{g}{l}} \tag{11}$$

Substituting equation (10) into equation (9) results in:

$$x_0 = \frac{d}{t}\sqrt{\frac{l}{g}} \tag{12}$$

Where *t* is measured, and *d, l,* and *g* are constants. The value of *g* will be considered 9.81ms^{-2} for the purposes of this investigation.

3. EXPERIMENTAL CONSIDERATIONS:

a. Variables Involved:

<u>Independent:</u>

In this investigation, this is the oscillation number (N). This is relatively straightforward to measure, considering the index appropriated by the datalogger used.

<u>Dependent:</u>

The dependent variable examined is the amplitude of oscillation (x_0), measured in *m*.

This is calculated from the directly measured time taken for the bob to move past the equilibrium position (*t*/s). This is recorded using a photogate timer measuring how long a beam is blocked, correct to 0.000001s (6 decimal places). This data is logged using a Pasco GLX.

<u>Controlled:</u>

- **Initial Amplitude of Release:**
  While the exponential equation does not have inclusions for the effect of amplitude on the decay rate, this is still controlled.
    - The velocity of the bob is dependent on the amplitude – at higher velocities, however, the linear proportionality between F_{drag} and *v* ceases to be valid, as a consequence of fluid flow becoming increasingly turbulent – *citation needed*. Hence, this impacts the validity of the

exponential relationship. Data supporting this can be found in the appendix.

This was selected to be a small amplitude - 5° - for the reason that this ensures a low velocity, as well as the difficulty faced in ensuring strictly 2-dimensional motion at higher amplitudes – as decay occurred, small lateral displacements caused significant circular motion.

- **Shape of bob used:** This impacts the drag force experienced by the bob – for instance, a more aerodynamic bob would experience lesser. A spherical bob was chosen to maximize the applicability of Stokes' Law.

- **Length of string:** Maintaining this is important due to its effects on the angular frequency, which in turn impacts the calculated value of x_0.

- **Mass of bob used:** This is a consequence of the assumption that viscous drag is the dominant force. Variation of the bob mass would mean that at heavier masses, the amount of drag force due to dry friction at the point of suspension would be increased. Friction is directly proportional to the normal force between two surfaces, and increased mass would result in the string exerting greater force on the suspending rod, thereby increasing the frictional force. While it is uncertain that these variations would be detectable, controlling it improves the experiment nevertheless.

- **Manner of release:** Variation in this would result in different degrees of rotational motion of the bob, and a changing amount of potential energy converted to rotational kinetic energy. This would impact the precision of the experiment in calculating the value of x_0. The manner chosen was using a scale to position the bob carefully at the required displacement, aligning it with the axis of oscillation, and dropping the scale to facilitate release.

b. Key Assumptions:

1. The **drag force does not vary** with amplitude. As mentioned above, this effect is considerable at larger amplitudes, and would result in significant deviation from the exponential model being considered as oscillations progressed. However, this is negligible at the amplitudes being considered (citation).
2. The **time period can be considered constant.** From an analytical perspective,
3. The **rotational energy is insignificant compared to the kinetic energy.** The energy comes both from the potential energy of the bob at initial release, as well as torsion in the string. The portion of the effect due to the potential energy would affect the validity of equation (6), and hence is minimized. Minimal rotation still exists, but this is considered negligible.
4. **Exponential decay** is the trend followed by the energy, and hence amplitude of the pendulum. While theory does support this, it is impossible to prove that this is the case by experiment. It can only be demonstrated with some uncertainty – as a consequence, this is an important assumption when linearization is being performed.

a. Experimental Design:

A simple pendulum was constructed as depicted in the figures below:

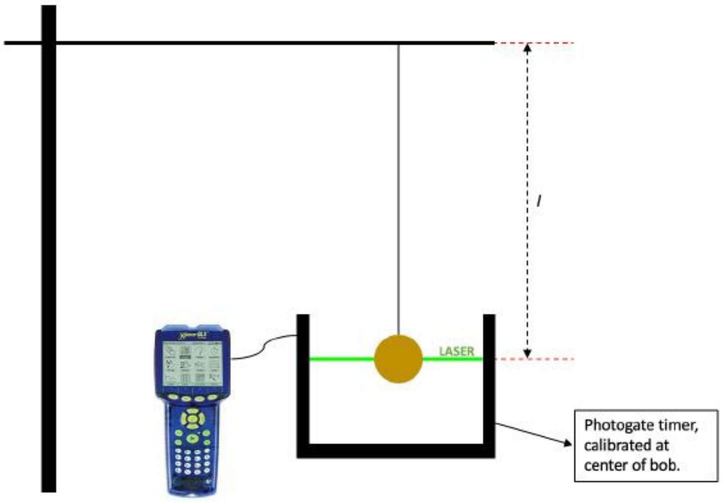


Fig. 1 – A side view of the apparatus

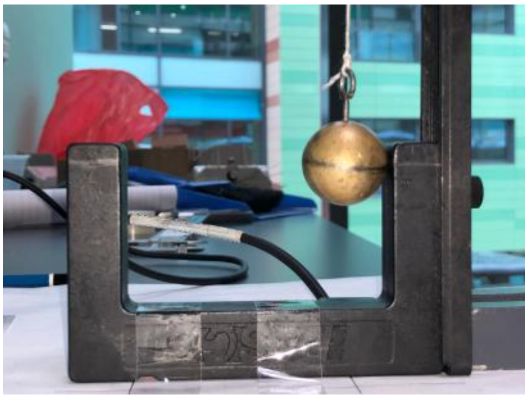


Fig. 2 – An image depicting the vertically calibrated bob (against the photogate). It was horizontally shifted for conducting oscillations.

Experimental Values:

l = 0.739m $\pm$ 0.0005m
d = 0.0248m $\pm$ 0.00005m (measured using Vernier Caliper)

The value of length *l* is arbitrary due to the fact that adjustments were required in order to position the bob precisely at the center-point of the photogate in order to ensure reliable readings. This was done using pencil marks on the photogate level and through the lateral diameter of the bob, and evaluation of the alignment by eye (*Fig 2.).*

In order to accurately determine that the angle of release, in addition to the protractor, the lateral displacement was calculated using the angle and the length *l*, and marked on a scale on the ground. However, this was difficult to accurately perform, and resulted in systematic error from set to set. Nevertheless, considering the assumption that the damping constant does not change with the amplitude of oscillation, this can be ignored.

Due to software incompatibility of the datalogger, a video was taken of readings being generated as the oscillations occurred, and this was manually entered into the Microsoft Excel software used for analysis. The time blocked was measured for 100 oscillations [200 values generated as movement in both directions is examined]. This was done because the uncertainty in the time recorded by the photogate is 0.000001s, which does not account for the significant variations in the data observed (see Fig. 4). As a consequence, multiple values, assuming no change in *b,* reduce the effect of random error. This is increased by 5 repetitions performed. The totality of 1000 readings generated was considered sufficient for the experiment.

5. **DATA & ANALYSIS:**

a. Raw Data:

Due to the overwhelming amount of data, the raw and processed data that will be presented is only for N = 1, 10, 25, 50, 75, and 100. However, all entered into a spreadsheet, and all relevant analysis was performed directly using it. Hence, extended analysis involving trendlines and damping constants is performed with data not entirely shown.

Table 1 presents the directly observed dependent variable, *t/s*, for above mentioned values of N, for all 5 trials.

Oscillation #(N)	**Time in Gate (t)/s**				
	T1	**T2**	**T3**	**T4**	**T5**
1	0.091362	0.086746	0.086830	0.085148	0.098339
10	0.094112	0.092281	0.091354	0.088772	0.103210
25	0.101707	0.099140	0.098259	0.095863	0.110321
50	0.115021	0.112030	0.111267	0.109016	0.123528
75	0.129883	0.126236	0.126785	0.124023	0.137192
100	0.146271	0.141541	0.145782	0.141434	0.154465

Table 1 – Raw Data for select values of N

When different trials were done, each differed systematically due to difficulty in precisely determining the angle of release, which had to be controlled. The expected initial blocked time can be determined using equation 7 – the analytical formula for this is the following (full derivation in Appendix 2):

$$t = \frac{d}{\sqrt{gl}\theta_0} \qquad (13)$$

Where θ_0 is in radians. When *l* = 0.739m, *d* = 0.0248m, and $\theta_0 = 5°$:

$$t_0 = \frac{0.0248}{\sqrt{9.81 \times 0.739} \times 5\left(\frac{\pi}{180}\right)} = 0.105547s$$

The initial values for the trials are in row 1 of Table 1. Their difference from t_0 does not present enough data to argue that there is clear systematic bias in one direction in due to the experiment, or to quantify this bias, but clearly this is a source of error. In order to correct for this, the closest value to $0.105547s$ was found for each trial and the numbering of oscillations was adjusted for N = 0 to begin with these. This reduced the total number of shared oscillations between trials to 58.5.

Consequently, the data presented is at the arbitrary points N = 0, 10, 20, 30, 40, 50, 58.5. This also enables the average t (t_{ave}/s) to be presented, as phase shift is corrected. Here,

Oscillation (N)	**Time in Gate (t)/s**					
	T1	**T2**	**T3**	**T4**	**T5**	t_{ave}/s
1	0.105499	0.105644	0.105446	0.105453	0.105385	0.105485
10	0.110352	0.111084	0.110916	0.110947	0.110321	0.110724
20	0.116623	0.116501	0.116699	0.116287	0.115425	0.116307
30	0.122787	0.121704	0.123001	0.122589	0.120743	0.122165
40	0.128372	0.127701	0.129425	0.129608	0.126350	0.128291
50	0.135010	0.133911	0.135803	0.136475	0.132309	0.134702
59.5	0.139847	0.139160	0.142715	0.141434	0.137192	0.140070

Table 2 – Corrected values of t.

b. Processed Data

Using equation (11), values of x_0 were calculated for all values of N. 3 significant figures are used due to the multiplication of *d* and $\sqrt{l}$ involved, which are of 3 significant figures each. Average values of *t* are used in processing.

$$x_0 = 0.0248\sqrt{\frac{0.739}{9.81}} \times \frac{1}{t} = \frac{0.00681}{t}\ m$$

Oscillation (N)	t_{ave}/s	$\frac{1}{t}/s^{-1}$	**Amplitude (x_0)/m**
1	0.105485	9.48	0.0646
10	0.110724	9.03	0.0615
20	0.116307	8.60	0.0586
30	0.122165	8.19	0.0557
40	0.128291	7.79	0.0531
50	0.134702	7.42	0.0506
58.5	0.140070	7.14	0.0486

Table 3 – Processing of t to $\boldsymbol{x_0}$

c. Propagation of Uncertainties

Since the independent variable is discrete, no error results there.

In the case of the dependent variable, based on equation (12), fractional uncertainties are added as follows:

$$\frac{\Delta x_0}{x_0} = \frac{\Delta t}{t} + \frac{\Delta d}{d} + \frac{\Delta l}{2l} \tag{14}$$

Given that *t* is measured correct to 0.000001s, the values of $\frac{\Delta t}{t}$ are on the order of 10^{-5}, which is negligible compared to uncertainty from other measurements.

$$\frac{\Delta d}{d} = \frac{0.00005}{0.0248} = 0.002016 \ldots \approx 0.00202$$

$$\frac{\Delta l}{2l} = \frac{0.0005}{0.739} = 0.006765 \ldots \approx 0.00077$$

In the case of Δt, given that *t* is measured correct to 0.000001s, the values of $\frac{\Delta t}{t}$ are on the comparatively negligible order of 10^{-5}. Instead, the method used is taking the standard deviation of values of *t* from all 5 trials, for each N to 58.5, and finding the average of all these values. Hence:

$$\Delta t = \frac{\Sigma(Standard\ Deviation)}{117} = 0.0090514 \approx 0.00905 \tag{15}$$

$\frac{\Delta t}{t}$ is then dependent on each value of t_{ave}. Hence:

$$\frac{\Delta x_0}{x_0} = 0.00077 + 0.00202 + \frac{0.00905}{t} = 0.00279 + \frac{0.00905}{t} \tag{16}$$

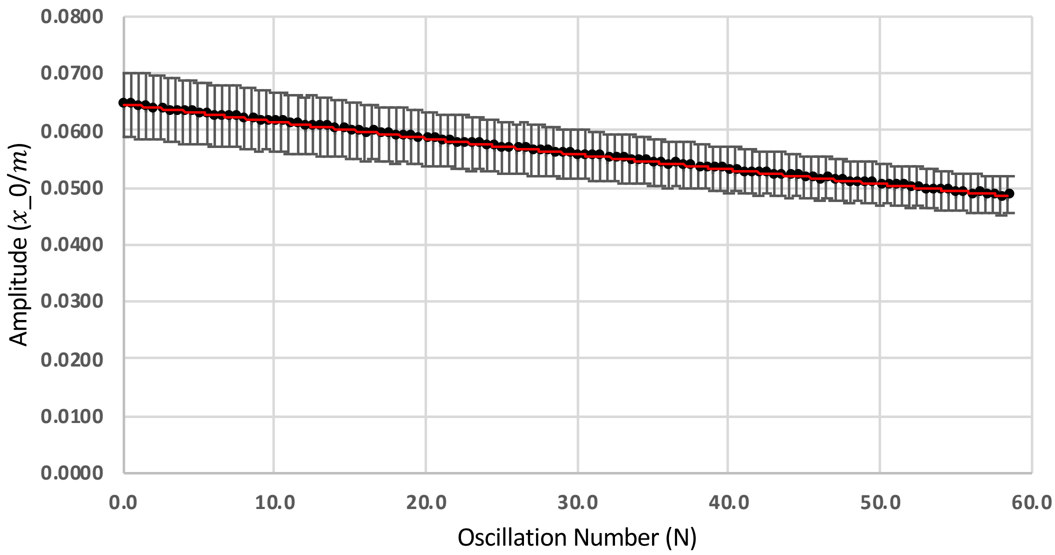
The calculation of $\frac{\Delta x_0}{x_0}$ and Δx_0 for N = 0, 10, 20, 30, 40, 50, and 58.5 are below:

Oscillation (N)	Amplitude (x_0)/m	$\frac{\Delta x_0}{x_0}$	$\Delta x_0/m$
1	0.0646	0.0886	0.0057
10	0.0615	0.0845	0.0052
20	0.0586	0.0806	0.0047
30	0.0557	0.0769	0.0043
40	0.0531	0.0733	0.0039
50	0.0506	0.0700	0.0035
58.5	0.0486	0.0674	0.0033

Table 4 – Uncertainties in $\boldsymbol{x_0}$

d. Analysis

The graph of Amplitude (x_0/m), derived from t_{ave}/s, against Oscillation Number (N), for all measured values, is shown below:



Graph 1 – Amplitude against Oscillation Number

The exponential decay trendline in red passes through all error bars of a relatively low uncertainty ranging from approximately 4.86% to 6.46%, and has a high R^2 value of 0.9996, indicating a strong decay fit. The equation is:

$$x_0 = 0.0647e^{-0.00489N} \quad (17)$$
$$b = 0.00489$$

However, the R^2 alone is insufficient to validate an exponential fit. When a linear fit is applied to the same data set, it too appears to pass through all error bars and has a lower, but still very high R^2 value of 0.9984. This discrepancy is addressed in the evaluation. In order to address it more reliably, under the assumption of exponential decay, the graph is linearized and standard error is calculated and compared to that of a linear fit of x_0 against N.

e. Extended Analysis

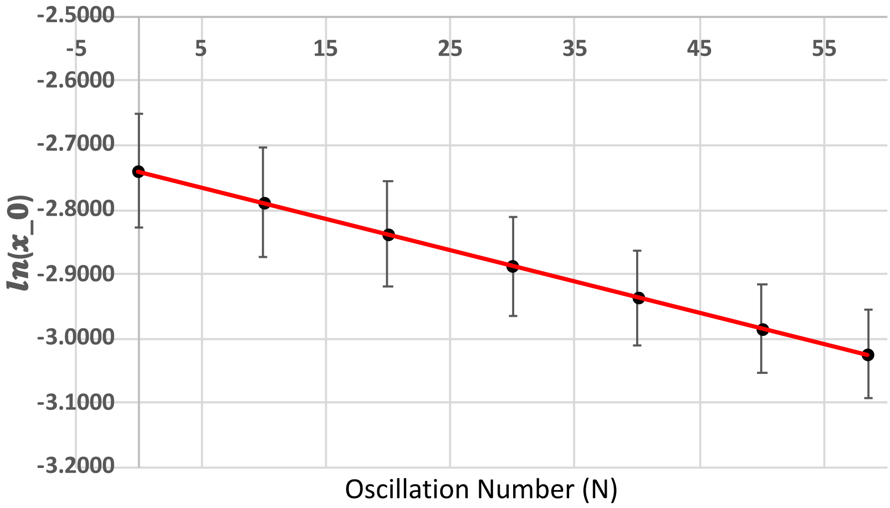
Linearization is carried out by taking the logarithm of both sides of equation (5). Hence:

$$\ln(x_0) = \ln(A) - bN$$

Oscillation (N)	**Amplitude (x_0)/m**	$\boldsymbol{ln(x_0)}$	$\Delta(\boldsymbol{ln(x_0)})$
1	0.0646	-2.74	0.089
10	0.0615	-2.79	0.085
20	0.0586	-2.84	0.081
30	0.0557	-2.89	0.077
40	0.0531	-2.94	0.073
50	0.0506	-2.98	0.070
58.5	0.0486	-3.02	0.067

Table 5 – Logarithm Table

To avoid a clutter of results, only points in Table 5 are graphed below:



Graph 2 – $\boldsymbol{ln(x_0)}$ against N

As seen here, the linear trendline passes almost perfectly through the points, and well within the error bars. The equation of the line for the entirety of the dataset, as expected from the exponential fit, is:

$$\ln(x_0) = -0.00489N - 2.74 \tag{18}$$

f. Standard Error Comparison

The standard error for an arbitrary variable *y* is calculated as follows:

$$SE = \sqrt{\frac{\Sigma(y_{predicted} - y_{real})^2}{n}}$$

This was carried out for the expected linear fit of $ln(x_0)$ against N (exponential model), as well as the unexpectedly high R^2 carrying fit of x_0/m against N (linear model). The predicted values were calculated according to the fit parameters provided by Excel. Here, *n* in both cases equals 117.

Percentage standard error is used for valid comparison. For %SE, the value of $\sqrt{(y_{predicted} - y_{real})^2}$ was computed for each value of N, and the corresponding values of x_0 and $ln(x_0)$ were used.

Model	**Standard Error**	**Percentage Standard Error (%)**
Exponential Model	0.000149	0.609
Linear Model	0.0000147	3.099

Table 6 – Standard Error Comparisons

The percentage standard error for the linear model is 5.09 times greater than that for the exponential model, which provides evidence for the fact that the exponential model is a far better fit for damping as a whole.

6. CONCLUSION & EVALUATION OF RESULTS

In answering the question "*how does the amplitude of an oscillating pendulum vary with the number of oscillations that have taken place?*", the **conclusion** drawn from this investigation is that a decay of the exponential nature is observed.

The observed competitiveness of the R^2 in a linear model is likely caused by the low damping ratio. As a result, the exponential decay in the first step is rather small. In order to validate this further, it would be advisable to conduct this same experiment with either of the following, in order to increase the value of *b*:

- A **more viscous fluid** – this increases resistant force per oscillation, of which b is a function
- A **larger ball** – a greater radius would also result in a more significant drag force, as per Stokes' law.

As indicated in Section 3d, the uncertainties calculated are dependent more on statistical deviation among samples and those arising from constants rather than the direct uncertainty in the dependent variable. This signifies that there exist other sources of error, that may include:

- **Lateral motion of the bob** – potential energy is converted to this kinetic energy too
- **Rotational Energy** – this might be more significant than assumed
- **Varying amplitudes** – the effect this has might be correctly neglected as per the assumption – however, this might have effects on the other sources of error, impacting the validity of the assumptions made.
- **Drag due to string** – this is not necessarily compliant with the Stokes' law model, and hence cannot be included in the exponential decay. As shown by Mohazzabi (et al)[2], this is a non-negligible effect.
- **Dry, Structural Damping** – these effects are less impactful since the string is tied below the rod – due to heating effects in the string itself, there might be additional non-negligible damping produced or general distortion of the oscillations.

A significant limitation in this experiment was that imposed by the lack of compatibility between the computer software for the Pasco GLX, and the operating system. As a consequence, the 1000 readings were manually recorded, each to 6 decimal places. This was an incredibly time-consuming process, as was the calibration procedures in aligning the photogate with the bob as closely as possible, to ensure no significant bias in the experiment that might impact the validity of the readings. If this was not the case, more readings could have taken with a greater reliability in the use of statistical processes used in determining how accurate the exponential decay model is.

[2] Mohazzabi, Pirooz, and Siva P. Shankar. "Damping of a simple pendulum due to drag on its string." Journal of Applied Mathematics and Physics 5.01 (2016): 122.

Further studies would include the isolation of the individual variables affecting the quality of the experiment, and examining this effect. This would allow the development of a quantitative correction formula that can be applied to the readings obtained, in order to get a set of data that is more reliable for the assessment of the validity of the exponential decay model. In addition, the effect of the initial amplitude on the damping constant was demonstrated by Aggarwal (et al)[3] can be further explored and mathematically quantified – this is an interesting function due to the fact that the assumption of an exponential rise in *b* as amplitude is changed would mean that the drag force varies with the velocity raised to a function of velocity itself, which is an interesting mathematical problem.

Applications of a comprehensive understanding of damping in a pendulum extend not only to situations where a pendulum is in use, such as tuned mass dampers in buildings, but to a variety of harmonic oscillators in general. Damping is of concern in particle physics, Josephson Junctions (coupling device for 2 superconductors), resonant circuits and much more. The system of the pendulum, arguably more complex than the rest when the multitude of variables that might result in damping are concerned, is a strong basis for the analysis of these effects in other harmonic oscillators in general, and an exploration of this might lead to engineering relevance that is as of yet undiscovered.

7. BIBLIOGRAPHY:

1. Aggarwal, Neha, Nitin Verma, and P. Arun. "Simple pendulum revisited." *European journal of physics* 26.3 (2005): 517.

2. “Dropping the Ball (Slowly).” *Stokes' Law*, galileo.phys.virginia.edu/classes/152.mf1i.spring02/Stokes_Law.htm

3. Mohazzabi, Pirooz, and Siva P. Shankar. "Damping of a simple pendulum due to drag on its string." *Journal of Applied Mathematics and Physics* 5.01 (2016): 122.

[3] Aggarwal, Neha, Nitin Verma, and P. Arun. "Simple pendulum revisited." European journal of physics 26.3 (2005): 517.

APPENDIX

1. Proof of solution to differential equation

The differential equation is as follows:

$$m\ddot{x} = kx - c\dot{x} \qquad (1)$$

This can be re-expressed using different terms, using the following substitutions:

$$\omega_0^2 = \frac{k}{m} \qquad (2)$$

ω_0^2, here, represents the square of the initial angular frequency.

$$\gamma = \frac{c}{m} \qquad (3)$$

Considering this, equation (1) is rewritten as:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \qquad (4)$$

Assuming a complex solution, z, equation is then expressed as:

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = 0 \qquad (5)$$

The complex solution is expressed as:

$$z = Ae^{i(pt+\alpha)} \qquad (6)$$

This is because this the exponential part of this function represents the equivalent of a trigonometric function with an imaginary sine and a real cosine.

When the derivatives of z, with respect to t, are taken and substituted into equation (5), it can be expressed as:

$$(-p^2 + i\gamma p + \omega_0^2)z = 0 \qquad (7)$$

Given this, either of $(-p^2 + i\gamma p + \omega_0^2)$ or z must equal 0. z, however, cannot equal 0 since it represents the solution to the displacement over time, and this changes in an oscillating system. Hence:

$$(-p^2 + i\gamma p + \omega_0^2) = 0. \qquad (8)$$

In this case, the real and imaginary parts of the equation are required to independently add up to 0. However, $i\gamma p \neq 0$, for the following reasons. Due to the definition of γ in equation (3), if it were to equal 0, *c*, the drag coefficient, would be equal to 0 too. However, given that this would imply no damping at all, this isn't possible. Given that in equation (6), *p* is part of the oscillatory component of the solution for *z*, this cannot equal 0 either. Hence, it can be deductively reasoned that

p must be complex in order for equation (9) to be satisfied. Therefore, p is defined as follows:

$$p = n + is \qquad (10)$$

And:

$$p^2 = n^2 - s^2 + 2ins \qquad (11)$$

When these are now substituted into equation (9), the following equation forms:

$$-n^2 + s^2 - 2ins + i\gamma n - \gamma s + \omega_0^2 = 0 \qquad (12)$$

The imaginary part of this equation is:

$$-2ins + \ i\gamma n = 0 \qquad (13)$$

Hence, $s = \frac{\gamma}{2}$. This, when substituted into the real part of equation (12), yields the following:

$$-n^2 + \frac{\gamma^2}{4} - \frac{\gamma^2}{2} + \omega_0^2 = 0 \qquad (14)$$

This means n can be expressed as:

$$n^2 = \omega_0^2 - \frac{\gamma^2}{4} \qquad (15)$$

Reverting to the proof of solution, first, equation (10) is substituted into equation (6):

$$z = Ae^{i(nt+ist+\alpha)} \qquad (16)$$

$$z = Ae^{-st}e^{i(nt+\alpha)} \qquad (17)$$

$$z = Ae^{-\frac{\gamma}{2}t}e^{i(nt+\alpha)} \qquad (17)$$

Clearly, n is representative now of the angular frequency of oscillation ω. This means the relation between the damped frequency of oscillation and the initial frequency of oscillation is as follows:

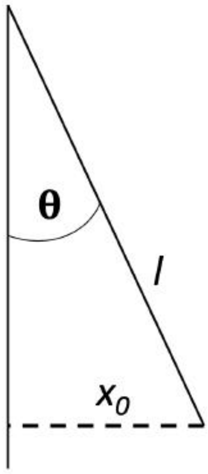
$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} \qquad (18)$$

Finally, the real part of the solution in equation (17) is the following:

$$x = Ae^{-\frac{\gamma}{2}t}\cos{(\omega t + \alpha)} \qquad (19)$$

For the sake of simplicity, this study considers the constant $\frac{\gamma}{2}$ as b. It is also worth noting that relative to the multiplied $\frac{\gamma}{2}$ in the case of the reduction of amplitude, the reduction in equation (18) is negligible, and hence unaccounted for.

2. Derivation of Equation (9) – Section 3(a)



Deducing based on the above figure:

$$x_0 = l \sin\theta \quad (1)$$

Based on the small angle approximation:

$$x_0 = l \sin\theta \approx l\theta \quad (2)$$

It is known from Section 3(b):

$$\frac{1}{2}mv_{\max}{}^2 = \frac{1}{2}m\omega^2 x_0{}^2 \quad (3)$$

This is rearranged, and the substitution $v_{max} = \frac{d}{t}$ is made. Hence:

$$\frac{d^2}{t^2} = \omega^2 x_0{}^2 \quad (4)$$

Substituting equation (2), and $\omega^2 = \frac{g}{l}$:

$$\frac{d^2}{t^2} = \frac{g}{l} \times l^2 \times \theta^2 \quad (5)$$

Re-arranging the following:

$$t = \frac{d}{\sqrt{gl}\theta_0} \quad (6)$$